# Wrapped M5-branes, Chern-Simons theory and holography


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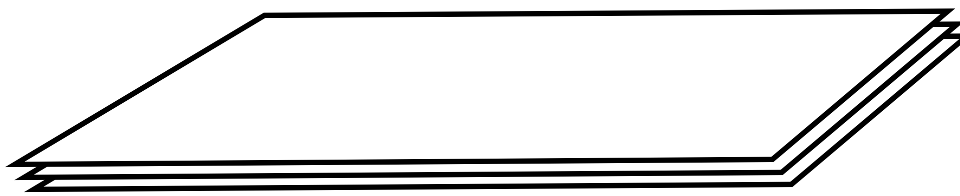
# Introduction

“3d-3d relation” : 3-manifolds $M$ ⟷ 3d $\mathcal{N}=2$ SCFTs $T[M]$

Interpretation from M5-branes : 6d= 3d + 3d

6d $A_{N-1}$ (2,0) theory



$N$ $M5s$

$A_{N-1}$ (2,0) theory *on* $M$ ⟶ $T_N[M]$

New insights on
- Physics of 3d $\mathcal{N}=2$ SCFTs
- Mathematics on 3-manifold

# Localization on 3d $\mathcal{N}=2$ SCFTs

[Kim ,Kapustin, Willet, Yakkov, Jafferis,
Hama, Hosomichi, Lee, Yokoyama, Imamura,…: '09 ~]

## $3d$ $\mathcal{N}=2$ theory

Gauge group G, Chiral matters $\Phi$ in R,

$CS$ interactions $\vec{\mathrm{k}}$, superpotential $\mathrm{W}(\Phi)$

## Localization on $B = S^2 \times S^1$, $S_b^3 / \mathbb{Z}_k$

$$Z = \int [d\Phi]_B \exp\left(iS\left[\Phi;(G,R,\vec{k},W(\Phi))\right]\right)$$

$$\xrightarrow{\text{Localized}} \int d\phi_0 e^{iS[\phi_0]} Z^{1-loop}[\phi_0] \quad \text{(finite dimensional integration)}$$

# State-integral in Complex CS theory

[Dimoft,Gukov,Lenells,Zagier, Garoufalidis, Gabella, Goncharov…: '09 ~]

## $SL(N)$ CS theory on 3-manifold $M$

$$S_{CS} = \frac{1}{2\hbar} CS[\mathcal{A}] + \frac{1}{2\tilde{\hbar}} CS[\overline{\mathcal{A}}], \quad \frac{4\pi}{\hbar} = k + \sigma, \ \frac{4\pi}{\tilde{\hbar}} = k - \sigma \ \text{ with } \ k \in \mathbb{Z}, \sigma \in \mathbb{R} \text{ or } i\mathbb{R}$$

$$CS[\mathcal{A}] := \int_M \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}^3 .$$

## State-integral model

$$Z = \int [d\mathcal{A}]_M \exp\left(iS_{CS}[\mathcal{A}, \overline{\mathcal{A}}; k, \sigma]\right)$$

$$\longrightarrow \int dX \exp\left(\frac{1}{2\hbar} X \cdot B^{-1} A\, X + ..\right) \prod \psi_{\hbar}(X) \ \text{(finite dimensional integration)}$$

# 3d-3d relation

[Yamzaki-Terashiam, Dimoft-Gukov-Gaiotto, '11 ]

For $N$ and $M$ , $\exists$ 3d $\mathcal{N}=2$ SCFT $T_N[M]$

$$Z[3d\ T_N[M]\ \text{theory on}\ B = S_b^3/\mathbb{Z}_k]$$

$$= Z[\ SL(N,\mathbb{C})\ CS\ \text{theory with level}\ (k,\sigma)\ \text{on}\ M\ ]$$

$$\left(k = k\ , \sigma = k(1-b^2)/(1+b^2)\right)$$

$k=1,$ $\quad Z[3d\ T_N[M]\ \text{theory on}\ B = S_b^3]$

$$= Z[\ SL(N,\mathbb{C})\ CS\ \text{theory on}\ M\ ]$$

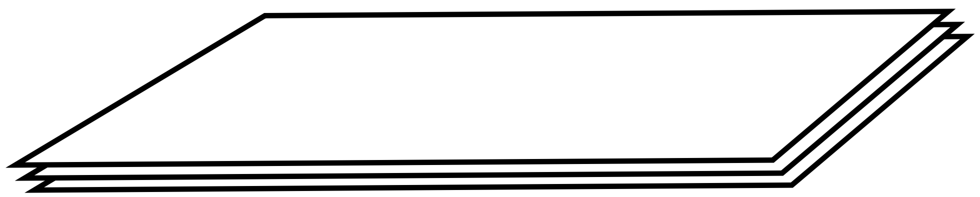
$$\left(\ \hbar = 4\pi/(k+\sigma) = 2\pi i b^2,\ \tilde{\hbar} = 4\pi/(k-\sigma) = 2\pi i b^{-2}\right)$$

# 3d-3d relation from M5

- How to explain the relation?

Hint ) 3d + 3d = 6d = dim(M5)

6d $A_{N-1}(2,0)$ theory



$N\ M5s$

$Z[A_{N-1}\ (2,0)\ \text{theory}\ on\ B \times M\ ]$

$Z[T_N[M\ ]\ \text{on}\ B]$

$Z[SL(N,\mathbb{C})\ CS\ \text{on}\ M\ ]$

[Jafferis-Cordova: '13]

- We can define $T_N[M\ ]$ as

$T_N[M\ ] : A_{N-1}\ (2,0)$ theory compactified along $M$

# Large N and holography

- $T_N[M]$ : $A_{N-1}$ (2,0) theory compactified along $M$

  $N\ M5s : \mathbb{R}^{1,2} \times M \longrightarrow 3d\ \mathcal{N}=2$ SCFT $T_N[M]$ on $\mathbb{R}^{1,2}$

- Gravity dual of $T_N[M]$ [Pernici,Sezgin : `85] [Gauntlett, Kim, Waldram : `00]

  $3d\ T_N[M]$ theory = M – theory on (Pernici-Sezgin AdS$_4$solution)

$$ds_{11}^2 = \frac{(1+\sin^2\theta)^{1/3}}{g^2}\left[ds^2(AdS_4) + ds^2(M) + \frac{1}{2}(d\theta^2 + \frac{\sin^2\theta}{1+\sin^2\theta}d\phi^2) + \frac{\cos^2\theta}{1+\sin^2\theta}d\tilde{\Omega}^2\right]$$

$g^3 \sim 1/N$

$3d\ T_N[M]$ theory on $B = S_b^3$

Holography ↔ M-theory on ‘Pernici-Sezgin AdS4’(b)

3d-3d relation ↔ $SL(N)$ $CS$ theory on $M$ , $\hbar = 2\pi i b^2$

# Large N and holography

3d $T_N[M]$ theory on $B = S_b^3$

Holography

3d-3d relation

M-theory on 'Pernici-Sezgin AdS4'(b)

$SL(N)$ $CS$ theory on $M$ , $\hbar = 2\pi i b^2$

$$\log\left|Z[T_N[M] \text{ theory on } S_b^3]\right|$$

$$= -\mathcal{F}^{\text{gravity}}(\text{'Pernici-Sezgin } AdS_4\text{'}(b))$$

$$= -\frac{\text{vol}(M)}{12\pi}(b+b^{-1})^2 N^3 + o(N^2)$$

# Large N and holography

$3d\ T_N[M]$ theory on $B = S_b^3$

Holography

3d-3d relation

M-theory on ‘Pernici-Sezgin AdS4’(b)

$SL(N)$ $CS$ theory on $M$ , $\hbar = 2\pi i b^2$

$$\log\left|Z[T_N[M] \text{ theory on } S_b^3]\right| = -\frac{\text{vol}(M)}{12\pi}(b+b^{-1})^2 N^3 + o(N^2)$$

$$= -\frac{i}{6\hbar}\text{vol}(M)N^3 - \frac{\text{vol}(M)}{6\pi}N^3 + \hbar\frac{i\,\text{vol}(M)}{24\pi^2}N^3 + o(N^2)$$

$(\hbar = 2\pi i b^2)$

$$\log(Z[T_N[M] \text{ theory on } S_b^3])$$

$$= \log(Z[SL(N)\ CS \text{ theory on } M])$$

$$\xrightarrow{\hbar\to 0} \frac{1}{\hbar}S_0[N;M] + S_1[N;M] + \hbar S_2[N;M] + \hbar^2 S$$

- It leads to a mathematical conjecture

$$\lim_{N\to\infty}\frac{1}{N^3}\text{Im}[S_0] = -\frac{1}{6}\text{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\text{Re}[S_1] = -\frac{1}{6\pi}\text{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\text{Im}[S_2] = \frac{1}{24\pi^2}\text{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\text{Re}[S_{2j-1}] = \lim_{N\to\infty}\frac{1}{N^3}\text{Im}[S_{2j}] = 0\ (j\geq 2)\,,$$

- Conjecture

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}] = -\frac{1}{6}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}] = \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}] = 0\ (j\geq 2)\,,$$

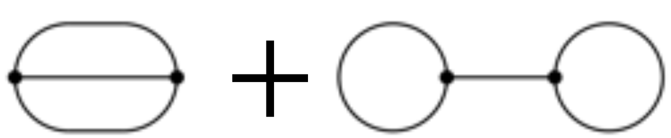
$S_n^{(\mathrm{conj})}$ n-loop contribution to the perturbative free-enery around a saddle point $\mathcal{A}_N^{(\mathrm{conj})}$

$\mathcal{A}_N^{(\mathrm{conj})} = \rho_N((\text{spin-connection}) + i(\text{dreibein}))\,,\ \rho_N : N-$dim irreducible representation of $SL(2)$

$$S_0^{(\mathrm{conj})} = 1/2\, CS[\mathcal{A}^{(\mathrm{conj})}]\ \text{(classical part)}\,,$$

$$S_1^{(\mathrm{conj})} = \frac{1}{2}\log\frac{[\det{}'\Delta_0(\mathcal{A}^{(\mathrm{conj})})]^{3/2}}{[\det{}'\Delta_1(\mathcal{A}^{(\mathrm{conj})})]^{1/2}}\,,$$

$\Delta_n(\mathcal{A}^{(\mathrm{conj})})$ : Laplacian on $sl(N)-$valued n-form twisted *by* $\mathcal{A}^{(\mathrm{conj})}$

$S_2^{(\alpha)} =$ 

- Conjecture

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}]=-\frac{1}{6}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}]=-\frac{1}{6\pi}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}]=\frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}]=\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}]=0\ (j\geq 2)\,,$$

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For S0 (classical part), from direct computation

$$S_0^{(\mathrm{conj})}[N]=\frac{1}{2}CS[\mathcal{A}_N^{(\mathrm{conj})}]=\frac{1}{6}N(N^2-1)(-i\mathrm{vol}(M)+....)$$

For S1 (1-loop part), it can be proven using a mathematical theorem

[P. Menal-Ferrer and J. Porti : `11]

For higher Sn , we don' t have analytic proof yet.

$+\hbar S_2[N,M]+\hbar^2 S_3[N,M]+\ldots$

- # Conjecture

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}]=-\frac{1}{6}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}]=-\frac{1}{6\pi}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}]=\frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}]=\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}]=0\ (j\geq 2)\,,$$

- # Numerical check

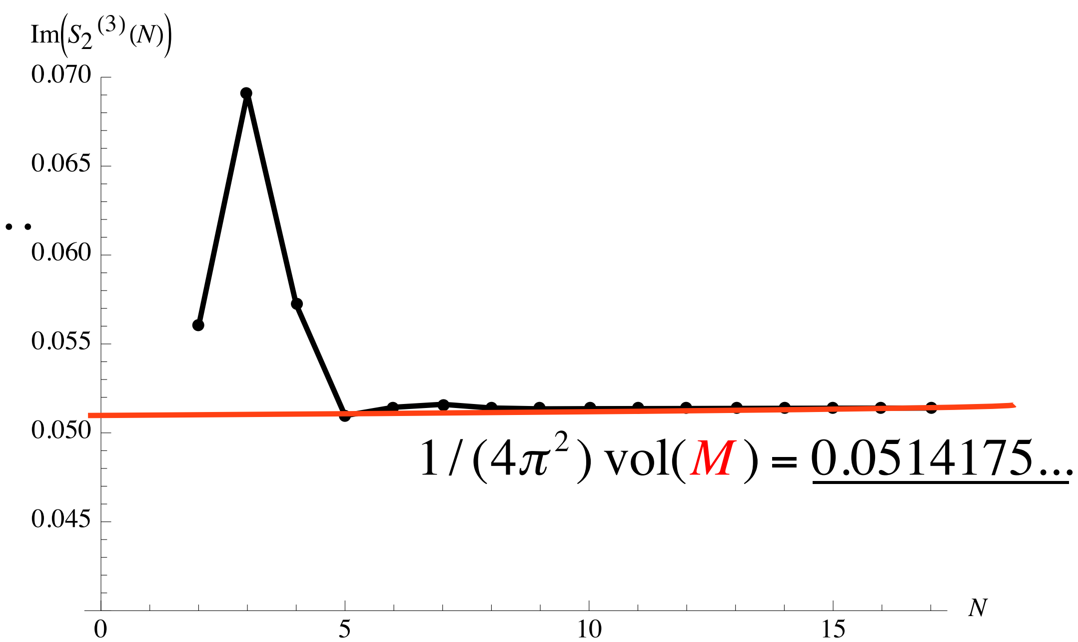
$M=S^3\setminus \mathbf{4}_1$ , $\mathrm{vol}(M)=2\mathrm{Im}[\mathrm{Li}_2(e^{-i\pi/3})]=2.02988..$

$\mathrm{Im}[S_2^{(\mathrm{conj})}(N)]=\{0.0882063,\ 0.289984,\ 0.618779,\ 1.13059,\ \ldots$

$N=2$ 3 4 5

$\mathrm{Im}[S_2'''(N)]=\{0.0560005, 0.0690888, 0.0572193,$
$0.0509708, 0.0514399, 0.0516042, \ldots$

$$\mathrm{Im}[S_2^{(\mathrm{conj})}(N)]\sim\frac{1}{24\pi^2}\mathrm{vol}(M)N^3$$



$$+\hbar S_2[N, M] + \hbar^2 S_3[N, M] + \dots$$

- Conjecture

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}] = -\frac{1}{6}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

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- Numerical check

$$M = S^3\backslash \mathbf{4}_1\ ,\quad \mathrm{vol}(M) = 2\mathrm{Im}[\mathrm{Li}_2(e^{-i\pi/3})] = 2.02988..$$

$$\mathrm{Im}[S_2^{(\mathrm{conj})}(N)] \sim \frac{1}{24\pi^2}\mathrm{vol}(M)N^3$$

$$S_3^{(\mathrm{conj})}(N) = \{-0.0185185, -0.0362503, -0.0425853, -0.0396434, -0.0348546, -0.0312819, -0.0284423, -0.0260191\} \ll N^3$$

# Summary

- 3d $T_N[M]$ theory

$N$ $M5s$ : $\mathbb{R}^{1,2} \times M$ $\longrightarrow$ 3d $\mathcal{N}=2$ SCFT $T_N[M]$ on $\mathbb{R}^{1,2}$

- Holography/3d-3d relation

3d $T_N[M]$ theory on $B=S^3_b$

Holography

3d-3d relation

M-theory on 'Pernici-Sezgin AdS4'(b)

$SL(N)$ CS theory on $M$ , $\hbar = 2\pi i b^2$

$$\lim_{N\to\infty} \frac{1}{N^3} \text{Im}[S_0^{(\text{conj})}] = -\frac{1}{6}\text{vol}(M)\,,\ \lim_{N\to\infty} \frac{1}{N^3} \text{Re}[S_1^{(\text{conj})}] = -\frac{1}{6\pi}\text{vol}(M)\,,\ \lim_{N\to\infty} \frac{1}{N^3} \text{Im}[S_2^{(\text{conj})}] = \frac{1}{24\pi^2}\text{vol}(M)$$

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- Further direction [Work in progress]

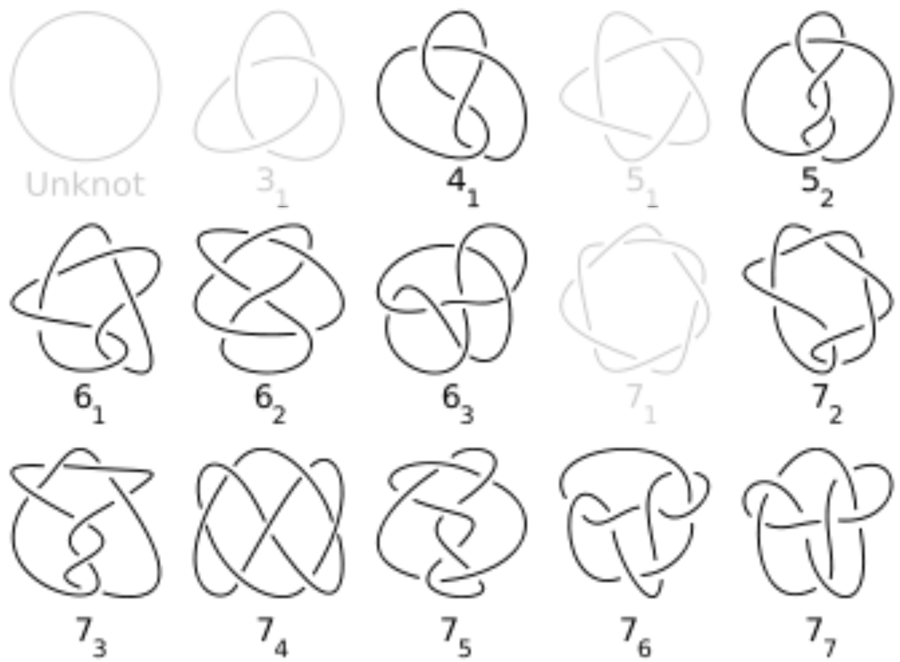
Defects : Codimension 2, 4 defects in 6d (2,0) theory

# 3d-3d relation for knot complements

- $k=1$

$Z[3d\ T_N[M]$ theory on $B=S_b^3] = Z[\ SL(N,\mathbb{C})\ CS$ theory on $M]\ \left(\hbar = 2\pi i b^2, \tilde{\hbar} = 2\pi i b^{-2}\right)$

- $M$ = knot complements $= S^3 \setminus \mathcal{K}$ ,

$\mathcal{K} =$ 


$M = S^3 \setminus N_{\mathcal{K}}$ , $N_{\mathcal{K}}$ : Tubular neighorhood of knot $\mathcal{K}$ (topologically solid torus)

$cf)\ M_2 = S^2 \setminus$ (punctures)

$S^3 \setminus \mathbf{4}_1 = S^3 \setminus$ 

$\partial M = S^1 \times S^1 = \{$meridian,longitudinal$\}$

| $3d\ T_N[S^3 \setminus \mathcal{K}]$ theory | $SL(N)\ CS$ theory on $M = S^3 \setminus \mathcal{K}$ |
|---|---|
| Rank of flavor symmetry | 1/2 dim $P_N(\partial M)$ ( $P_N(\partial M) := \{SL(N)$ flat-connections on $\partial M\}$ ) |
| $U(1)^{N-1} \longrightarrow SU(N)$ flavor sym | $P_N(\partial M) = \{\mathbf{m} = \mathrm{Hol}_{meridian}(\mathcal{A}), \mathbf{l} = \mathrm{Hol}_{longtude}(\mathcal{A}) : [\mathbf{m},\mathbf{l}] = 0\}$ |
| mass-parameter $\{m_i\}_{i=1,\ldots,N-1}$ | boundary meridian holonomy $\mathbf{m} \sim \begin{pmatrix} \exp(m_1) & 1 & 0 & 0 & \cdots \\ 0 & \exp(m_2) & 1 & 0 & \cdots \\ 0 & 0 & \exp(m_3) & 1 & \cdots \\ 0 & 0 \cdots & & \cdots & \cdots \end{pmatrix}$ |